TOPICS IN SET THEORY: Example Sheet 3 [1]

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge

Michaelmas 2014-2015 Dr Oren Kolman

1 Clubs

(i) Suppose $\kappa = cf(\kappa) > \aleph_0$. Show $U = \{\alpha < \kappa : \omega\alpha = \alpha\}$ and $V = \{\alpha < \kappa : \omega^\alpha = \alpha\}$ are club in κ. Give an example of two disjoint clubs in $\aleph_\omega$. Do there exist disjoint stationary subsets of $\aleph_1$ and of $\aleph_2$?

(ii) Suppose that A is club in $\kappa = cf(\kappa) > \aleph_0$ and $f : A \subseteq \kappa \to \kappa$ is a function. Prove that $B = \{\alpha \in A : (\forall \xi < \alpha)(f(\xi) < \alpha)\}$ is club in κ. Deduce that if $\lambda < cf(\kappa)$ and F is a family of λ many functions from A into κ, then the set $C = \{\alpha \in A : (\forall f \in F)(\forall \xi < \alpha)(f(\xi) < \alpha)\}$ contains a club in κ. In other words, the ordinals which are closed under members of F contain a club.

(iii) Suppose $D \subseteq \kappa = cf(\kappa) > \aleph_0$. Show that D is a club of κ if and only if D is the range of a continuous strictly increasing function $f : \kappa \to \kappa$.

(iv) Prove that if $\delta > cf(\delta) > \aleph_0$, then there is a club E of δ such that no member of E is a regular cardinal. [Hint. Try the range of a continuous function $f : \kappa \to \{\alpha \in \kappa : \alpha > cf(\kappa)\}$.]

(v) Optional. Suppose $\mathbb{M}$ is a τ-structure with domain κ where τ is a vocabulary of cardinality less than $\kappa = cf(\kappa) > \aleph_0$. Show that $\{\delta \in \kappa : \mathbb{M} \restriction \delta$ is an elementary substructure of $\mathbb{M}\}$ is a club of κ. [Hint. This assumes some first-order model theory: use Skolem functions and/or the elementary chain theorem.]

2 Non-stationary sets

Suppose $\kappa = cf(\kappa) > \aleph_0$, and X_α is non-stationary in κ for $\alpha < \kappa$.

(i) Show $\bigcup_{\alpha<\kappa}(X_\alpha \setminus (\alpha + 1))$ is non-stationary in κ.

(ii) If $\{X_\alpha : \alpha < \kappa\}$ is pairwise disjoint, prove $X = \bigcup_{\alpha<\kappa} X_\alpha$ is stationary if and only if $B = \{min(X_\alpha) : \alpha < \kappa\}$ is stationary.

(iii) If $\{X_\alpha : \alpha < \kappa\}$ is pairwise disjoint, then there exists $a \in [\kappa]^\kappa$ such that $\bigcup_{\alpha\in a} X_\alpha$ is non-stationary.

3 Applications

(i) Suppose $f : \omega_1 \to \mathbb{R}$ is a continuous function, where ω_1 has the order topology. Prove $(\exists \alpha < \omega_1)(\forall \beta > \alpha)(f(\beta) = f(\alpha))$, i.e. f is eventually constant.

(ii) The ordinal ω_1 with the order topology is not a metrizable topological space.

(iii) Prove the following result of Shelah. Suppose S is a stationary subset of $\kappa = cf(\kappa) > \aleph_0$ and g and h are functions from S into λ such that $(\forall \xi \in S)(g(\xi) \neq h(\xi))$. Then there exists a stationary subset $S' \subseteq S$ such that:

$$\{g(\xi) : \xi \in S'\} \cap \{h(\zeta) : \zeta \in S'\} = \emptyset.$$

[1] Version 16/11/2014; comments, improvements and corrections will be much appreciated; please send to ok261@cam.ac.uk; rev. 14/12/2014.

[HINT. Note there is a stationary set $D_1 \subseteq S$ such that $(\forall \zeta, \eta \in D_1)((g(\zeta) < \zeta \Leftrightarrow g(\eta) < \eta) \wedge (h(\zeta) < \zeta \Leftrightarrow h(\eta) < \eta))$; apply Fodor's Lemma to find a stationary subset $D_2 \subseteq D_1$ such that if $g(\zeta) < \zeta$ for all $\zeta \in D_1$, then g is constant on D_2 and the same for h; now use Question 1 to obtain a club C closed under the functions g and h. Let $S' = D_2 \cap C$.]

4 CLUB FILTERS

A *filter* over a non-empty set I (or on $P(I)$, in $P(I)$, or sometimes on I) is a family $F \subseteq P(I)$ such that

(i) $\emptyset \notin F, I \in F$;
(ii) if $X, Y \in F$, then $X \cap Y \in F$;
(iii) if $X \in F, X \subseteq Y \subseteq I$, then $Y \in F$.

A filter F is *principal* if $F = \{X \in P(I) : Y \subseteq X\}$ for some $Y \in P(I)$; otherwise F is *non-principal.* A filter F over I is κ-*complete* if for every $\lambda < \kappa$ and $\{X_\alpha \in F : \alpha < \lambda\}$, $\bigcap_{\alpha<\lambda} X_\alpha \in F$. A filter F over κ is *normal* if for every $\{X_\alpha \in F : \alpha < \kappa\}$, the diagonal intersection $\triangle_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : (\forall \alpha < \xi)(\xi \in X_\alpha)\} \in F$.

(i) Suppose F is a normal filter over a cardinal κ, $S \subseteq \kappa, \kappa \setminus S \notin F$, and f is a regressive function on S. Show that there exists $X \subseteq S$ and $\gamma < \kappa$ such that $\kappa \setminus X \notin F$ and $(\forall \xi \in X)(f(\xi) = \gamma)$.

(ii) Suppose $\kappa \geq cf(\kappa) > \aleph_0$. The *club filter* on κ is the family $C_\kappa = \{X \in P(I) : X \supseteq C$ for some club C in $\kappa\}$. Note that C_κ is a filter.
  (a) Show that C_κ is $cf(\kappa)$-complete.
  (b) Prove that if $\kappa = cf(\kappa) > \aleph_0$, then C_κ is normal.

5 SPLITTING STATIONARY SETS AND SOLOVAY'S THEOREM

(i) Suppose $\kappa = cf(\kappa) > \aleph_0$. Prove that there exists a family $\{S_\alpha \subseteq \kappa : \alpha < \kappa\}$ of pairwise disjoint stationary sets such that $\kappa = \bigcup_{\alpha<\kappa} S_\alpha$. [HINT. Consider cases according as κ is a limit cardinal or a successor cardinal, using some of the elements of an Ulam matrix on κ in the latter case.]

(ii) OPTIONAL**: SOLOVAY'S THEOREM. Suppose S is a stationary subset of $\kappa = cf(\kappa) > \aleph_0$. Prove that there exists a family $\{S_\alpha : \alpha < \kappa\}$ of pairwise disjoint stationary sets such that $S = \bigcup_{\alpha<\kappa} S_\alpha$.

(iii) Suppose S is stationary in $\kappa = cf(\kappa) > \aleph_0$. Prove that there exists a family F of 2^κ stationary subsets of S such that if $A \neq B \in F$, then $A \setminus B$ and $B \setminus A$ are stationary in κ. [HINT. Part (ii).]

6 NON-STATIONARY IDEALS

An *ideal* over a non-empty set I is a family $N \subseteq P(I)$ such that

(i) $\emptyset \in N, I \notin N$;
(ii) if $X, Y \in N$, then $X \cup Y \in N$;

(iii) if $X \in N, Y \subseteq X \subseteq I$, then $Y \in F$.

An ideal is an ideal over $dom(I) = \bigcup I$. Let $N^+ = P(I) \setminus N$ be the family of N-non-negligible sets. The *dual filter* N^* of an ideal N is the filter $\{X \in P(I) : I \backslash X \in N\}$. An ideal N is κ-complete and normal if the corresponding dual filter N^* has these properties. The *dual ideal* F^* of a filter F is defined analogously: $F^* = \{X \in P(I) : I \backslash X \in F\}$. Clearly, for an ideal N and a filter F, $N^{**} = N; F^{**} = F$. For a cardinal κ, the *non-stationary ideal over* κ is the ideal $NS_\kappa = \{X \in P(I) : X \subseteq Y$ for some non-stationary subset $Y \subseteq \kappa\}$. So NS_κ^+ is the collection of stationary sets of κ. An ideal N is λ-*saturated* if for any $\{X_\alpha : \alpha < \lambda\} \subseteq N^+$ there exist $\beta < \gamma < \lambda$ such that $X_\beta \cap X_\gamma \in N^+$. Let $sat(N) = min\{\lambda : N$ is λ-saturated$\}$.

(i) Show $C_\kappa^* = NS_\kappa$.

(ii) Prove Ulam's theorem: there is no κ^+-saturated κ^+-complete ideal over κ^+.

(iii) Suppose that for some $\lambda < \kappa, N$ is a λ-saturated κ-complete ideal over κ. Determine whether κ has the tree property, i.e. whether every κ-tree has a cofinal branch. [Hint. WLOG, any candidate κ-tree $\mathbb{T}$ has domain $T = \kappa$; consider $D_\xi = \{\zeta \in \kappa : \xi <_{\mathbb{T}} \zeta\}$.]

7 Stationary Sets and a Variant of Fodor's Lemma

Let A be a set of cardinality $\kappa = cf(\kappa) > \aleph_0$. A κ-*filtration* of A is an indexed sequence $\{A_\alpha : \alpha < \kappa\}$ such that for all $\alpha, \beta < \kappa$

(a) $\mid A_\alpha \mid < \kappa$;

(b) $\alpha < \beta$ implies $A_\alpha \subseteq A_\beta$;

(c) $\delta \in lim(\kappa)$ implies $A_\delta = \cup\{A_\alpha : \alpha < \delta\}$;

(d) $A = \cup\{A_\alpha : \alpha < \kappa\}$.

(i) Suppose $\{A_\alpha : \alpha < \kappa\}$ and $\{B_\alpha : \alpha < \kappa\}$ are κ-filtrations of A. Show the set $\{\alpha \in \kappa : A_\alpha = B_\alpha\}$ is a club of κ.

(ii) Let $\{A_\alpha : \alpha < \kappa = cf(\kappa)\}$ be a κ-filtration of A. Prove there exists a club C of κ such that for all $\alpha \in C$, $\mid A_{\alpha^+} \setminus A_\alpha \mid = \mid \alpha^+ \setminus \alpha \mid$ where α^+ is the successor of α in C, i.e. $\alpha^+ = inf\{\beta \in C : \beta > \alpha\}$.

(iii) Suppose $\{A_\alpha : \alpha < \kappa\}$ is a κ-filtration of a set A of cardinality κ. Prove the following variant of Fodor's Lemma: if S is a stationary subset of κ and $f : S \to A$ is a function such that for all $\alpha \in S$, $f(\alpha) \in A_\alpha$, then there exists a stationary $S' \subseteq S$ such that $f \restriction S'$ is constant.

8 Clubs and Games

Let $W \subseteq [\omega_1]^{<\omega_1}$. Let G_W be the following game of length ω: players I and II take turns to choose countable ordinals $\alpha_0, \alpha_1, \ldots$; player I wins G_W if $\{\alpha_n : n \in \omega\} \in W$. Regarding each countable ordinal α as the set $\{\beta : \beta < \alpha\}$, show that player I has a winning strategy if and only if W contains a club of ω_1. Show that player II has a winning strategy if and only if the complement of W contains a club. Deduce that there are games G_W which are not determined, i.e. neither player has a winning strategy.

9 Weakly Compact Cardinals

(i) Let A be a set of cardinals such that for every regular cardinal $\lambda \in A, A \cap \lambda$ is not stationary in λ. Prove there exists an injective function g on A such that $(\forall \alpha \in A)(g(\alpha) < \alpha)$.

(ii) Suppose that κ is a *weakly compact cardinal*, i.e. κ is (strongly) inaccessible and there are no κ-Aronszajn trees (κ has the tree property). Prove that for every stationary subset S of κ, there is a regular cardinal $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ.

10 Kueker Clubs

Suppose $\kappa = cf(\kappa) > \aleph_0$. A *club on* $[A]^{<\kappa}$ is a family $C \subseteq [A]^{<\kappa}$ such that C is closed under unions of chains of length less than κ and $(\forall X \in [A]^{<\kappa})(\exists Y \in C)(X \subseteq Y)$. A set $S \subseteq [A]^{<\kappa}$ is *stationary in* $[A]^{<\kappa}$ if $S \cap C \neq \emptyset$ for every club C in $[A]^{<\kappa}$. Formulate and prove analogues of the standard results on clubs, stationary sets and regressive functions for the above definitions.

11 Prove that $\diamondsuit$ implies that there exists a family $\{Z_\alpha : \alpha < 2^{\aleph_1}\}$ such that:

(i) $\forall \alpha < 2^{\aleph_1}, Z_\alpha$ is a stationary subset of $\aleph_1$;

(ii) $\forall \alpha < \beta < 2^{\aleph_1}, Z_\alpha \cap Z_\beta$ is countable.

12 Inconsistent Bogus Diamonds

Let $\blacklozenge(1)$ be the assertion $(\exists \{A_\alpha \subseteq \alpha : \alpha < \omega_1\})(\forall X \subseteq \omega_1)(\{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ contains a club in $\omega_1)$. Let $\blacklozenge(2)$ be the assertion $(\exists \{A_\alpha \subseteq \alpha : \alpha < \omega_1\})(\forall X \subseteq \omega_1)($if X is stationary, then $\{\alpha \in X : X \cap \alpha = A_\alpha\} \neq \emptyset)$.

(i) Show $\blacklozenge(1)$ is false.

(ii) Show $\blacklozenge(2)$ is false.

13 Diamonds for Functions

Suppose that S is a stationary subset of $\lambda = cf(\lambda) > \aleph_0$. Let $\diamondsuit_\lambda(S)$ denote the assertion $(\exists \{A_\alpha \subseteq \alpha : \alpha \in S\})(\forall X \subseteq \lambda)(\{\alpha \in S : X \cap \alpha = A_\alpha\}$ is stationary in $\lambda)$. Let $\diamondsuit_\lambda$ mean $\diamondsuit_\lambda(\lambda)$; so in this notation $\diamondsuit$ is $\diamondsuit_{\omega_1}$.

Prove that $\diamondsuit_\lambda(S)$ is equivalent to the assertion: there exists $\{f_\alpha : \alpha \in S\}$ such that:

(i) $\forall \alpha \in S, f_\alpha : \alpha \to \alpha$ is a function;

(ii) $\forall f : \lambda \to \lambda, \{\alpha \in S : f \restriction \alpha =: f_\alpha\}$ is stationary in λ.

14 Diamond Equivalences

Let $\diamondsuit'$ denote the following assertion: there exists a family $\{E_\alpha : \alpha \in \omega_1\}$ such that:

(i) $\forall \alpha \in \omega_1, E_\alpha$ is a countable set of subsets of α;

(ii) $\forall X \subseteq \omega_1, \{\alpha \in \omega_1 : X \cap \alpha \in E_\alpha\}$ is stationary in ω_1.

Prove that $\diamondsuit'$ and $\diamondsuit$ are equivalent in ZFC.

15 Diamond and the Club Prediction Principle

(i) Show that $\diamondsuit$ implies $\clubsuit$.

(ii) Prove Devlin's theorem (1979): $\clubsuit + CH$ implies $\diamondsuit$. Deduce that $\diamondsuit$ is equivalent to $\clubsuit + CH$.

16 Stronger Diamonds

Let $\diamondsuit^+$ denote the assertion there exists $\{S_\alpha \subseteq P(\alpha) : \alpha \in \omega_1\}$ such that $\mid S_\alpha \mid \leq \aleph_0$ and $(\forall X \subseteq \omega_1)(\exists B \in [\omega_1]^{\omega_1})(\forall \alpha < \omega_1)(\alpha = sup(B \cap \alpha) \Rightarrow X \cap \alpha \in S_\alpha \wedge B \cap \alpha \in S_\alpha)$.

(i) Prove that $\diamondsuit^+$ implies $\diamondsuit'$.

(ii) Deduce that $\diamondsuit^+$ implies $\diamondsuit$.

Remark: Jensen proved that $\diamondsuit^+$ implies the Kurepa Hypothesis. This stronger diamond can be defined for the other uncountable cardinals κ and used to settle combinatorial hypotheses about κ-trees.

17 Optional. Ineffable cardinals and the κ-Kurepa Hypothesis

A cardinal κ is *ineffable* if $\kappa = cf(\kappa) > \aleph_0$ and whenever $f : [\kappa]^2 \to 2$ is a function, then there is a stationary set $S \subseteq \kappa$ such that $f \restriction S$ is constant, i.e. $(\exists \gamma)(\forall \xi \in S)(f(\xi) = \gamma)$.

(i) Let $\kappa = cf(\kappa) > \aleph_0$. Suppose $(*)$ holds: for every $\{A_\alpha \subseteq \alpha : \alpha < \kappa\}$, there exists $A \subseteq \kappa$ such that $\{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ is stationary in κ. Prove that κ is ineffable.

(ii) Prove that if κ is ineffable, then $(*)$ holds.

(iii) Prove the following result of Jensen and Kunen: if κ is ineffable, then the κ-Kurepa Hypothesis is false. [Hint. Do this by refuting the existence of a *κ-Kurepa family*, i.e. a family $F \subseteq P(\kappa)$ such that $\mid F \mid = \kappa^+$ and $(\forall \alpha < \kappa)(\{X \cap \alpha : X \in F\}$ has cardinality at most $\mid \alpha \mid + \aleph_0)$; it is a well-known short theorem that the existence of such a family is equivalent to that of a κ-Kurepa tree.]

18 (i) Suppose π is a bijection from $\lambda^+ \geq \aleph_1$ onto $\lambda \times \lambda^+$. Show there exists a club C of λ^+ such that for all $\delta \in C$, the restriction map $\pi \restriction \delta$ is a bijection from δ onto $\lambda \times \delta$.

(ii) For a cardinal λ and a set W, let $[W]^{\leq \lambda} = \{Y \subseteq W : \mid Y \mid \leq \lambda\}$. If $2^\lambda = \lambda^+$, let $\{X_\alpha : \alpha < \lambda^+\}$ be an enumeration of $[\lambda^+]^{\leq \lambda}$ and suppose $Z \subseteq \lambda^+$. Show that for some club C of λ^+, for all $\delta \in C$ there are arbitrarily large $\alpha < \delta$ such that for some $\beta < \delta, Z \cap \alpha = X_\beta$.

(iii) Suppose $cf(\delta) = \kappa > \aleph_0$ and h is a function from $dom(h) \supseteq \delta$ into κ. Prove that the following are equivalent:

(a) h is one-to-one on some club C of δ;
(b) h is strictly increasing on some club D of δ;
(c) $range(h \restriction S)$ is unbounded in κ for every stationary subset $S \subseteq \delta$.

Remark These propositions are the first steps of a very recent short proof by Peter Komjath of Shelah's theorem (see the reference below) that $2^\lambda = \lambda^+$ implies $\diamondsuit_{\lambda^+}$ for $\lambda \geq \aleph_1$; this is not the case for $\lambda = \aleph_0$ as CH does not imply $\diamondsuit$.

Open Research Problems.

(i) Assume that $\lambda = \lambda^{<\lambda} = 2^\mu$ is a regular limit cardinal. Determine whether $\diamondsuit_\lambda$ is a theorem of ZFC.
See: S. Shelah, Diamonds, Proc. Amer. Math. Soc. 138 (2010), 2151–2161 ; M. Zeman, Diamonds, GCH and weak square, Proc. Amer. Math. Soc. 138 (2010), 1853–1859.